Variance Lower Bound and Asymptotic Normality of Scrambled Geometric Nets

Kinjal Basu

Relevance Sciences, LinkedIn

Joint work with Rajarshi Mukherjee

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Overview

1. Introduction

2. Scrambled Geometric Nets

3. Lower Bound on Variance

4. Asymptotic Normality
The Problem

- Numerical integration
- Domain of interest: \( \mathcal{X}^s = \prod_{j=1}^s \mathcal{X}^{(j)} \), where each \( \mathcal{X}^{(j)} \subset \mathbb{R}^d \).
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\[
\mu = \frac{1}{\text{vol}(\mathcal{X}^s)} \int_{\mathcal{X}^s} f(x) \, dx
\]

by an equal weight rule

\[
\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} f(x_i),
\]

where \( x_i \) are the points generated by QMC or RQMC methods.
Motivation

Need to construct a confidence interval.

Asymptotic Distribution.


Matching lower bound for the variance.
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- Need to construct a confidence interval.
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- Matching lower bound for the variance.
Our Strategy for Non-Cubical Space $\mathcal{X}^s$

- Start with a $(t, m, s)$-net in base $b$ in $[0, 1]^s$.
- Introduce randomization via Scrambling Algorithm to get $u \in [0, 1]^s$.
- Apply a mapping $\phi$ such that $x_i = \phi(u_i) \in X$ (Scrambled geometric net).
- Equal weight rule $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} f(x_i)$.

Most interesting case: triangles, spherical triangles and discs.
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Previous Results

Lemma 1 (B. and Owen (2015b))

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Theorem 1 (B. and Owen (2015b))
There exists a constant \( C > 0 \) such that
\[
\text{Var}(\hat{\mu}_n) \leq C \frac{(\log n)^{s-1}}{n^{1+2/d}},
\]
under certain smoothness conditions on \( f \) and a sphericity constraint on the partitioning of \( \mathcal{X}^s \).
Main Results

Point set is a scrambled \((0, m, s)\) geometric net in base \(b\).

Theorem 2. (B. and Mukherjee (2016))
If \(f \in F_s\) and the partitioning of \(X_s\) satisfies an eigenvalue condition, then there exists a positive constant \(c\) such that
\[
\text{Var}(\hat{\mu}_n) \geq c \left( \log n \right)^{s-1} n^{1/2} + 2/d.
\]

(2)

Define,
\[
W = \hat{\mu}_n - \mu / \sqrt{\text{Var}(\hat{\mu}_n)}.
\]

Theorem 3. (B. and Mukherjee (2016))
Let \(b \geq \max(s, d, 2)\), \(f \in F_s\) and if (2) holds, then
\(W \to N(0, 1)\) in distribution as \(n \to \infty\).
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(continued on next page)
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Let \(b \geq \max(s, d, 2)\), \(f \in \mathcal{F}_s\) and if (2) holds, then \(W \to \mathcal{N}(0, 1)\) in distribution as \(n \to \infty\).
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Scrambled Geometric Nets - A simpler construction

Set of \( n = b^m \) points on the domain \( X \).

Fix a equal volume recursive partition in base \( b \) of the domain.

Put a point \( x_i \) uniformly at random within a cell of volume \( \frac{1}{b^m} \).

For example on \( T^2 \) using base \( b = 4, \)
Scrambled Geometric Nets - A simpler construction

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- For example on $T^2$ using base $b = 4$,
Splits on the Triangle

Figure: Splits of a triangle $\mathcal{X}$ for bases $b = 2, 3$ and $4$. The subtriangles $\mathcal{X}_j$ are labeled by the digit $j \in \mathbb{Z}_b$. 
Recursive Splits on the Triangle

Figure: The base $b$ splits from previous figure carried out to $k = 6$ or 3 or 4 levels.
Splitting on the Disc

Figure: A recursive binary equal area splitting of the unit disk, keeping the aspect ratio close to unity.
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Form of Variance

- Using Multiresolution Analysis of $L^2(\mathcal{X}^s)$,

\[
\text{Var}(\hat{\mu}_n) = \mathbb{E} \left( \left[ \frac{1}{n} \sum_{i=1}^{n} (f(x_i) - \mu) \right]^2 \right) \\
= \frac{1}{n} \sum_{|u|>0} \sum_{\kappa|u} \Gamma_{u,\kappa} \sigma_{u,\kappa}^2.
\]
Form of Variance

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$$= \frac{1}{n} \sum_{|u|>0} \sum_{\kappa|u} \Gamma_{u,\kappa} \sigma^2_{u,\kappa}.$$

where

$$\sigma^2_{u,\kappa} = \sum_{\tau} \sum_{\gamma,\gamma'} \langle f, \psi_{u\kappa\tau\gamma} \rangle \langle f, \psi_{u\kappa\tau\gamma'} \rangle \prod_{j \in u} \left( 1_{c_j = c'_j} - \frac{1}{b} \right).$$
Main Theorem on Lower Bound

**Theorem 2: B. and Mukherjee (2016)**

If \( f \in \mathcal{F}_s \) and an eigenvalue condition holds for the partitioning of the domain, then there exists a positive constant \( c \) such that

\[
\text{Var}(\hat{\mu}_n) \geq c \frac{(\log n)^{s-1}}{n^{1+2/d}},
\]

for all sufficiently large \( n \).
Smooth class of functions $\mathcal{F}_s$

Definition

A real-valued function $f$ on $X$ is smooth if for all $u \subseteq s$, 

$$\|\nabla u f(x) - \nabla u f(x^*)\| \leq B \|x - x^*\|^\beta$$

for some finite $B \geq 0$ and $\beta \in (0, 1]$ for all $x, x^* \in X$.

Definition

Define $\mathcal{F}_s$ as the class of all smooth functions $f$ on $X$ such that for all $u \subseteq s$, 

$$\|\int_X \nabla u f(x) \, dx\|^2 > 0.$$
Smooth class of functions $\mathcal{F}_s$

**Definition**

A real-valued function $f$ on $\mathcal{X}^s$ is smooth if for all $u \subseteq s$,

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\|\nabla^u f(x) - \nabla^u f(x^*)\| \leq B \|x - x^*\|^\beta
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**Definition**

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$$\left\| \int_{\mathcal{X}^s} \nabla^u f(x) \, dx \right\|^2 > 0.$$
Remember that,\[
\sigma^2_u, \kappa = \sum_{\tau} \sum_{\gamma, \gamma'} \langle f, \psi_{u, \kappa \tau \gamma} \rangle \langle f, \psi_{u, \kappa \tau \gamma'} \rangle \prod_{j \in u} (c_j = c_j' - 1 b_j).
\]
Eigenvalue Condition

Remember that,

$$\sigma_{u,\kappa}^2 = \sum_{\tau} \sum_{\gamma,\gamma'} \langle f, \psi_{u\kappa\tau\gamma} \rangle \langle f, \psi_{u\kappa\tau\gamma'} \rangle \prod_{j \in u} \left( 1_{c_j = c_j'} - \frac{1}{b} \right).$$
Define, \( A(k, t) = b - \sum_{c=0}^{\infty} (n_c - w)(n_c - w) \) for some \( \tilde{c} > 0 \).
Define,

\[ A^{(k,t)} = \sum_{c=0}^{b-1} (n_c - w)(n_c - w)^T \]
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\[ \lambda_1 (A^{(k,t)}) \geq \tilde{c} b^{-2k/d} \text{ for some } \tilde{c} > 0. \]
Some Examples
Some Examples

Using the above subdivision, $A(k,t) = b - k\left[2 - \frac{1}{2} - \frac{1}{2}\right]$

Thus, $\lambda_1(A(k,t)) = b - k/6$

If $X = [0,1]$, then $A(k,t) = b - 2k(b^2 - 1)(b^2 - 12)$.
Some Examples

Using the above subdivision,

\[ A^{(k, t)} = \frac{b^{-k}}{6} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \]
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Thus, \( \lambda_1 (A^{(k,t)}) = b^{-k}/6 \)

If \( \mathcal{X} = [0, 1] \), then \( A^{(k,t)} = b^{-2k} \left( \frac{b^2-1}{12b} \right) \).
$f(x, y) = x_1 x_2^2 - y_1^3 y_2^4$ on $T^2 \times T^2$
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Example continued

![Graph showing density plots for different m values and comparison with True N(0,1)]
Normal Q-Q Plots

\( m = 6 \)

\[ \begin{align*}
\text{Theoretical Quantiles} & \quad \text{Sample Quantiles} \\
\end{align*} \]

\( m = 7 \)

\[ \begin{align*}
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\end{align*} \]

\( m = 8 \)

\[ \begin{align*}
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\( m = 9 \)

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Proof technique of Asymptotic Normality

\[ W = \frac{\hat{\mu}_n - \mu}{\sqrt{\text{Var}(\hat{\mu}_n)}} \]

Create a \( \tilde{W} \) satisfying

\[ W - \tilde{W} = o_p(1) \]

Create an exchangeable pair \((\tilde{W}, \tilde{W}^*)\)

Based on Exchangeable Pair technique of Stein's Method, show that \( \tilde{W} \to N(0,1) \) in distribution as \( m \to \infty \).

Use Slutsky's Theorem.
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- Use Slutsky’s Theorem.
Confidence Intervals

- $f(x, y) = x_1 x_2^2 - y_1^3 y_2^4$ on $T^2 \times T^2$
Confidence Intervals

\[ f(x, y) = x_1 x_2 y_1 y_2 \exp(x_1 x_2 y_1 y_2) \text{ on } T^2 \times T^2 \]
Take Away Message

There are good variance bounds for the estimator $\hat{\mu}_n$ for large $n$. Asymptotically accurate confidence sets can be easily constructed. QMC Methods can give us a way out of the Cube.
There are good variance bounds for the estimator $\hat{\mu}_n$ for large $n$. 

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Thank you

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- Art Owen
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References: