Variance Lower Bound and Asymptotic Normality of Scrambled Geometric Nets

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Relevance Sciences,
LinkedIn

Joint work with Rajarshi Mukherjee

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Overview

The Problem

- Numerical integration
- Domain of interest : $\mathcal{X}^s = \prod_{j=1}^{s} \mathcal{X}^{(j)}$, where each $\mathcal{X}^{(j)} \subset \mathbb{R}^d$.

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by an equal weight rule

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} f(\boldsymbol{x}_i), \tag{1}$$

where $\boldsymbol{x}_i$ are the points generated by QMC or RQMC methods.

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- Matching lower bound for the variance.

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- Most interesting case : triangles, spherical triangles and discs.

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Theorem 1 (B. and Owen (2015b))

There exists a constant $C > 0$ such that

$$\mathrm{Var}(\hat{\mu}_n) \leq C\frac{(\log n)^{s-1}}{n^{1+2/d}},$$

under certain smoothness conditions on f and a sphericity constraint on the partitioning of $\mathcal{X}^s$.

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If $f \in \mathcal{F}_s$ and the partitioning of $\mathcal{X}^s$ satisfies an eigenvalue condition, then there exists a positive constant c such that

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Theorem 3. (B. and Mukherjee (2016))

Let $b \geqslant \max(s, d, 2)$, $f \in \mathcal{F}_s$ and if (2) holds, then $W \to \mathcal{N}(0, 1)$ in distribution as $n \to \infty$.

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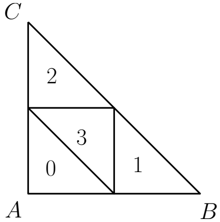
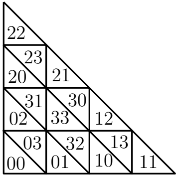
- For example on T^2 using base $b = 4$,



(a)



(b)

Splits on the Triangle

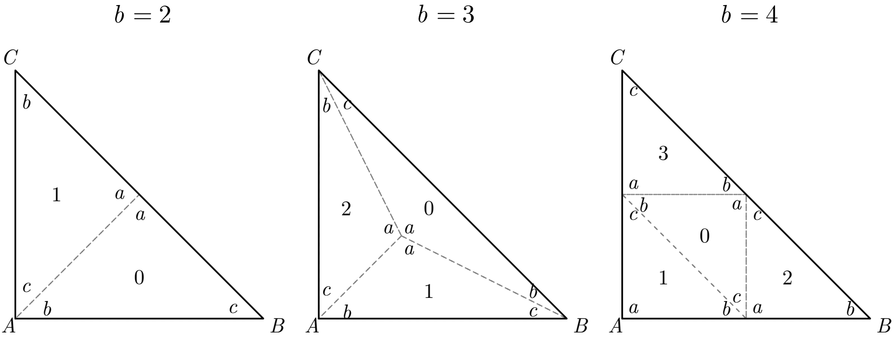


Figure: Splits of a triangle $\mathcal{X}$ for bases $b = 2$, 3 and 4. The subtriangles $\mathcal{X}_j$ are labeled by the digit $j \in \mathbb{Z}_b$.

Recursive Splits on the Triangle

2^6 Decomposition $\quad$ 3^3 Decomposition $\quad$ 4^3 Decomposition

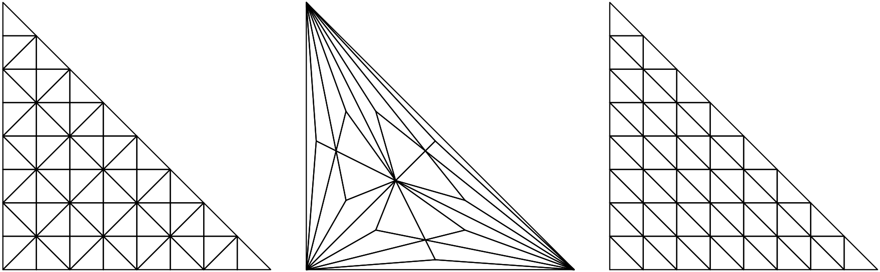


Figure: The base b splits from previous figure carried out to $k = 6$ or 3 or 4 levels.

Splitting on the Disc

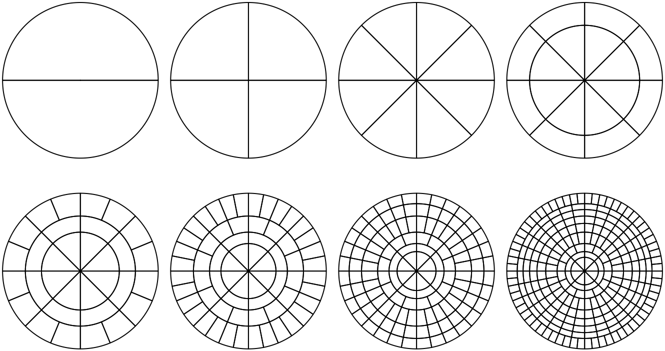


Figure: A recursive binary equal area splitting of the unit disk, keeping the aspect ratio close to unity.

Overview

Form of Variance

- Using Multiresolution Analysis of $L^2(\mathcal{X}^s)$,

$$
\begin{aligned}
\mathrm{Var}(\hat{\mu}_n) &= \mathbb{E}\left(\left[\frac{1}{n}\sum_{i=1}^{n}(f(\boldsymbol{x}_i) - \mu)\right]^2\right) \\
&= \frac{1}{n}\sum_{|u|>0}\sum_{\kappa|u}\Gamma_{u,\kappa}\sigma^2_{u,\kappa}.
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where

$$\sigma_{u,\kappa}^2 = \sum_{\tau}\sum_{\gamma,\gamma'}\langle f,\psi_{u\kappa\tau\gamma}\rangle\langle f,\psi_{u\kappa\tau\gamma'}\rangle\prod_{j\in u}\left(\mathbb{1}_{c_j=c_j'}-\frac{1}{b}\right).$$

Main Theorem on Lower Bound

Theorem 2: B. and Mukherjee (2016)

If $f \in \mathcal{F}_s$ and an eigenvalue condition holds for the partitioning of the domain, then there exists a positive constant c such that

$$\mathrm{Var}(\hat{\mu}_n) \geqslant c \frac{(\log n)^{s-1}}{n^{1+2/d}},$$

for all sufficiently large n.

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Definition

A real-valued function f on $\mathcal{X}^s$ is smooth if for all $u \subseteq s$,

$$\|\nabla^u f(\boldsymbol{x}) - \nabla^u f(\boldsymbol{x}^*)\| \leq B \|\boldsymbol{x} - \boldsymbol{x}^*\|^\beta$$

for some finite $B \geq 0$ and $\beta \in (0, 1]$ for all $\boldsymbol{x}, \boldsymbol{x}^* \in \mathcal{X}^s$.

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Definition

Define $\mathcal{F}_s$ as the class of all smooth functions f on $\mathcal{X}^s$ such that for all $u \subseteq s$,

$$\left\| \int_{\mathcal{X}^s} \nabla^u f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right\|^2 > 0.$$

Eigenvalue Condition

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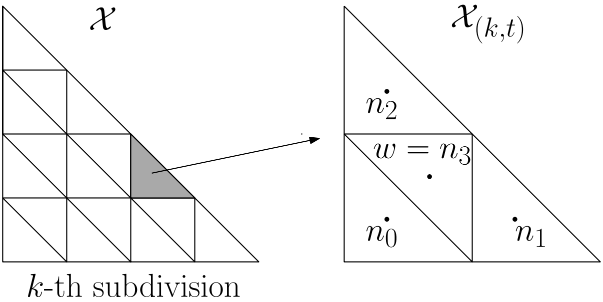
Remember that,

$$\sigma_{u,\kappa}^2 = \sum_{\tau} \sum_{\gamma,\gamma'} \langle f, \psi_{u\kappa\tau\gamma} \rangle \langle f, \psi_{u\kappa\tau\gamma'} \rangle \prod_{j \in u} \left(\mathbb{1}_{c_j = c_j'} - \frac{1}{b} \right).$$

Eigenvalue Condition



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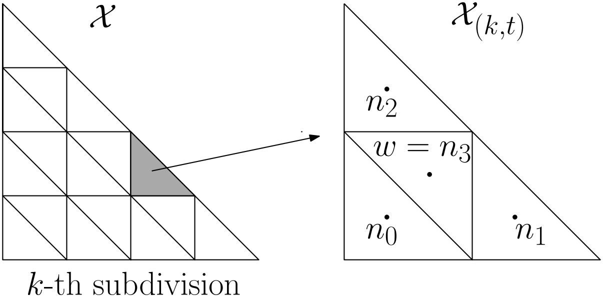

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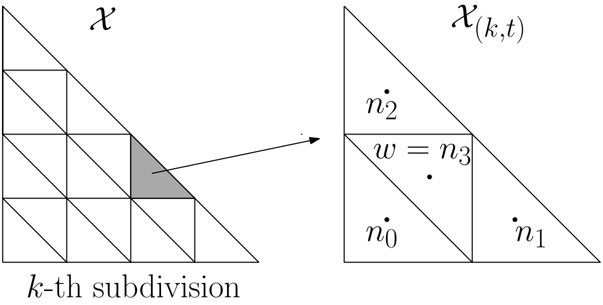

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- $\lambda_1 \left(A^{(k,t)} \right) \geq \tilde{c} b^{-2k/d}$ for some $\tilde{c} > 0$.

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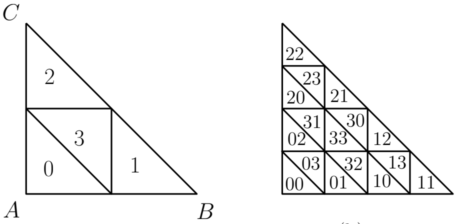

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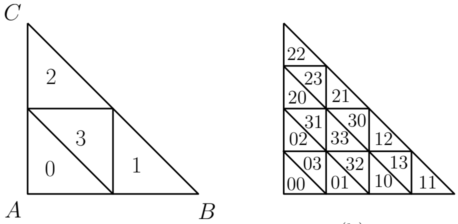

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$$A^{(k,t)} = \frac{b^{-k}}{6} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

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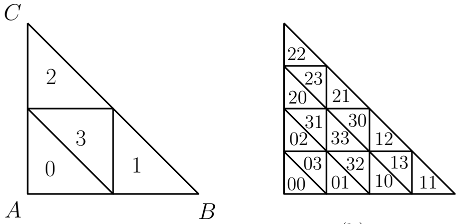

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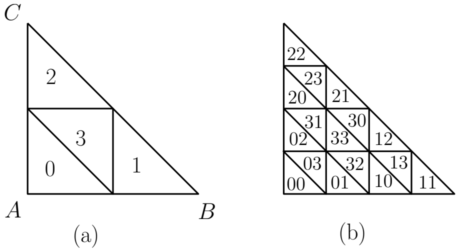

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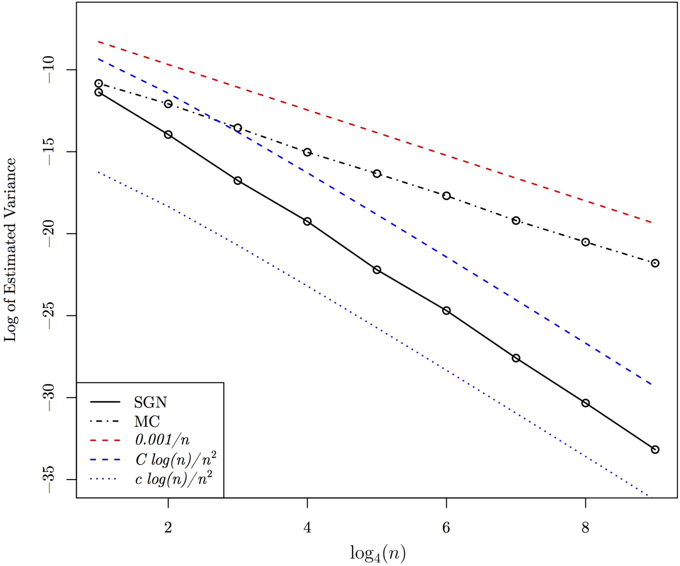
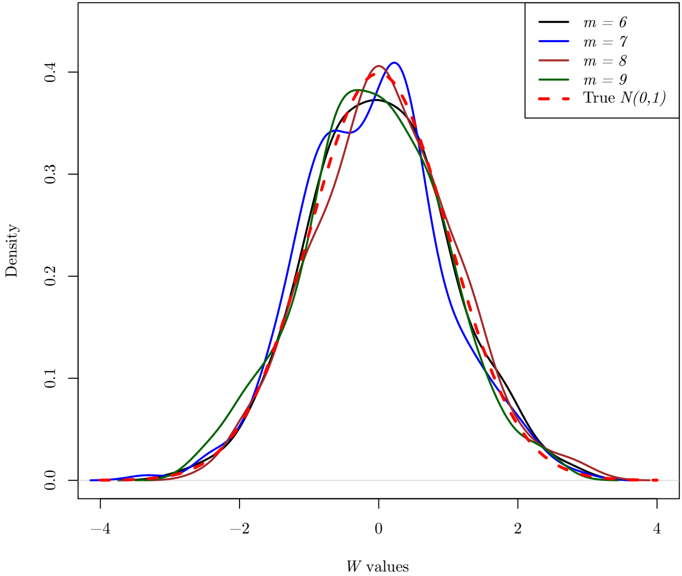
- Thus, $\lambda_1 \left(A^{(k,t)} \right) = b^{-k}/6$
- If $\mathcal{X} = [0, 1]$, then $A^{(k,t)} = b^{-2k} \left(\frac{b^2-1}{12b} \right)$.

$f(\boldsymbol{x}, \boldsymbol{y}) = x_1 x_2^2 - y_1^3 y_2^4$ on $T^2 \times T^2$

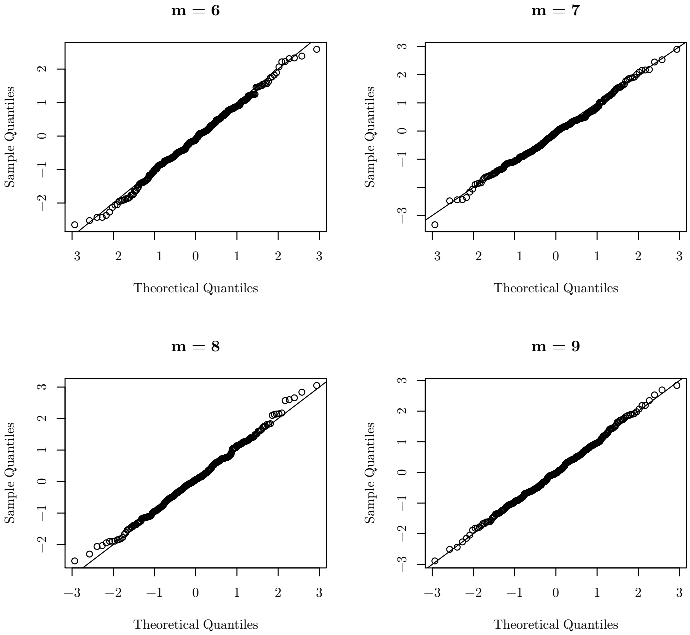


Overview

Example continued



Normal Q-Q Plots



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- Use Slutsky's Theorem.

Confidence Intervals

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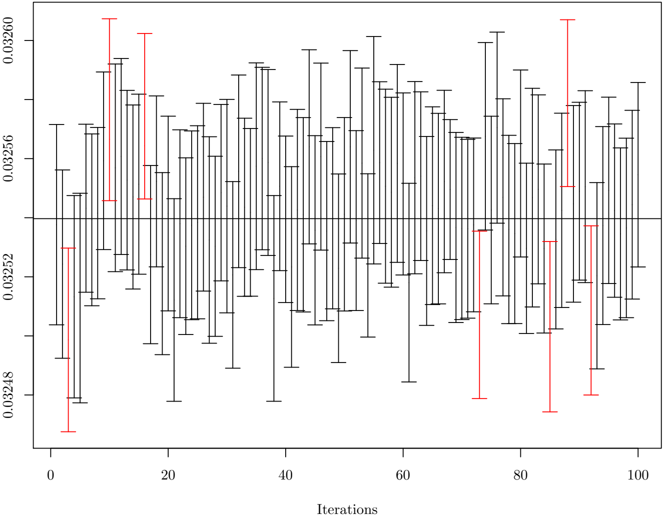
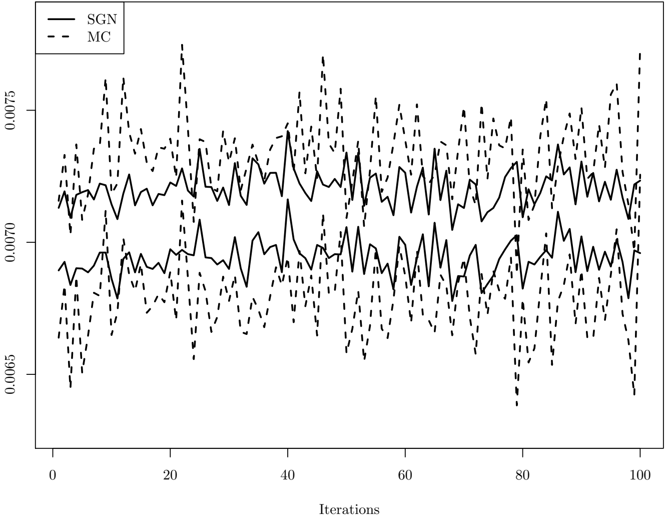


Confidence Intervals

- $f(\boldsymbol{x}, \boldsymbol{y}) = x_1 x_2 y_1 y_2 \exp(x_1 x_2 y_1 y_2)$ on $T^2 \times T^2$



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- There are good variance bounds for the estimator $\hat{\mu}_n$ for large n.
- Asymptotically accurate confidence sets can be easily constructed.
- QMC Methods can give us a way out of the Cube.

Thank you

- The organizers
- Co-author Rajarshi Mukherjee
- Art Owen
- NSF Grant DMS-1407397

References:

- Basu, K and Mukherjee, R. (2016) Asymptotic normality of scrambled geometric net quadrature. *The Annals of Statistics.* To Appear.
- Basu, K and Owen, A. (2015) Scrambled geometric net integration over general product spaces. *Foundations of Computational Mathematics.* In Press.